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Haraux-Weissler型方程式の正値球対称解について

On the Positve Radial Solutions to the Haraux-Weissler Equation

早稲田大学 理工学部　廣瀬 宗光
Waseda University　Munemitsu Hirose

1. Introduction

The aim of this talk is to investigate the structure of positive radial solutions to

(1.1)
$$\Delta u+\frac{1}{2}x\cdot\nabla u+\lambda u+|u|^{p-1}u=0,\ x\in\mathrm{R}^n,$$

where $p>1$, $n\geq 3$ and $\lambda\geq 0$. Since we are interested in radial solutions (i.e., $u=u(r)$ with $r=|x|$), we will study the following initial value problem

(IVP)
$$\begin{cases} u_{rr}+\dfrac{n-1}{r}u_r+\dfrac{r}{2}u_r+\lambda u+|u|^{p-1}u=0,\ r>0, \\ u(0)=\alpha>0. \end{cases}$$

Equation (1.1) comes from the study of a semilinear heat equation of the form

(1.2)
$$f_t-\Delta f-|f|^{p-1}f=0,\ (t,x)\in(0,\infty)\times\mathrm{R}^n.$$

When we discuss the following function, which is called a *self-similar solution*,

$$f(t,x):=t^{-1/(p-1)}u(x/\sqrt{t}),$$

it can be seen that f satisfies (1.2) if and only if u satisfies (1.1) with $\lambda=1/(p-1)$.

In Section 3, it will be shown that (IVP) has a unique solution $u(r)\in C^2([0,\infty))$ with $u_r(0)=0$, which is denoted by $u(r;\alpha)$. Moreover, if we define $z:=\inf\{r>0\ ;\ u(r;\alpha)=0\}$, then $u(r;\alpha)$ is decreasing in $[0,z)$. By the decreasing property of $u(r;\alpha)$, we can classify solutions of (IVP) in the following manner:

(i) $u(r;\alpha)$ is a *crossing solution* if $0 < z < +\infty$,

(ii) $u(r;\alpha)$ is a *decaying solution* if $z = +\infty$, i.e. $u(r;\alpha) > 0$ in $[0,\infty)$.

These terminologies are used by Yanagida and Yotsutani [YY1].

Many authors have studied (IVP). Weissler [W1] has proved that, if $\lambda \geq n/2$, then $u(r;\alpha)$ is a crossing solution for every $\alpha > 0$. For $0 < \lambda < n/2$, the critical exponent $p = (n+2)/(n-2)$ is important. Set $L := \lim_{r\to\infty} r^{2\lambda} u(r;\alpha)$. In the supercritical case $p \geq (n+2)/(n-2)$, Atkinson and Peletier [AP] and Peletier, Terman and Weissler [PTW] have proved that, if $0 < \lambda \leq \max\{1, n/4\}$, then $u(r;\alpha)$ is a decaying solution with $0 < L < +\infty$ for every $\alpha > 0$. Especially in the critical case $p = (n+2)/(n-2)$, Escobedo and Kavian [EK] have got the following result; if $\max\{1, n/4\} < \lambda < n/2$, then there exists a decaying solution with $L = 0$, i.e.,

$$u(r;\alpha) = C\exp(-r^2/4) r^{2\lambda-n}\left[1 + O(r^{-2})\right] \text{ as } r \to \infty,$$

where C is a positive constant. In the subcritical case $1 < p < (n+2)/(n-2)$, Weissler [W1] has proved that, if $\lambda > 0$, then $u(r;\alpha)$ is a crossing solution for sufficiently large α. Moreover, Haraux and Weissler [HW] have given an interesting result. Put

$$\alpha_* := \inf\{\alpha > 0 \,;\, u(r;\alpha) \text{ is a crossing solution}\}.$$

If $\lambda > 1/2(p-1)$ and $\lambda < n/2$, then $0 < \alpha_* < +\infty$ and $u(r;\alpha_*)$ is a decaying solution with $L = 0$. Moreover, $u(r;\alpha)$ is a decaying solution with $0 < L < +\infty$ for sufficiently small α.

Although we have picked up a part of known results, it seems that there are no works about the structure of solutions to (IVP) with $\lambda = 0$, and that the complete information for the structure of solutions to (IVP) with $\lambda > 0$ has not known. In this paper, we will show the structure of *positive* radial solutions to (IVP) with $\lambda = 0$, using the classification theorem by Yanagida and Yotsutani (see Section 4). Moreover, we will apply the same argument to (IVP) with $\lambda = 1$, and give more detailed information than the result in [HW].

2. Main Results

Our problem is to decide whether each $u(r;\alpha)$ is a crossing solution or a decaying solution when initial value α moves from 0 to $+\infty$. In the case $\lambda = 0$, we obtain the following result.

Theorem 1. Let $\lambda = 0$.

(i) If $p \geq (n+2)/(n-2)$, then $u(r;\alpha)$ is a decaying solution for every $\alpha > 0$.

(ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_0 such that $u(r;\alpha)$ is a decaying solution for every $\alpha \in (0,\alpha_0]$ and a crossing solution for every $\alpha \in (\alpha_0,\infty)$. Moreover, $u(r;\alpha_0)$ is the most rapidly decaying solution among decaying solutions such that

(2.1) $$u(r;\alpha_0) = O\left(r^{-n}\exp(-r^2/4)\right) \text{ as } r \to \infty.$$

In [YY1], Yanagida and Yotsutani have studied the structure of positive radial solutions to the Lane-Emden equation

(2.2) $$\Delta u + u^p = 0, \quad x \in \mathrm{R}^n.$$

A fundamental difference to the structure of positive radial solutions between (1.1) with $\lambda = 0$ and (2.2) appears in the subcritical case $1 < p < (n+2)/(n-2)$ because every positive radial solution to (2.2) is a crossing solution.

In the case $\lambda = 1$, we can show a similar result to the case $\lambda = 0$.

Theorem 2. Let $\lambda = 1$.

(i) If $p \geq (n+2)/(n-2)$, then $u(r;\alpha)$ is a decaying solution for every $\alpha > 0$.

(ii) If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_1 such that $u(r;\alpha)$ is a decaying solution for every $\alpha \in (0,\alpha_1]$ and a crossing solution for every $\alpha \in (\alpha_1,\infty)$. Moreover, $u(r;\alpha_1)$ is the most rapidly decaying solution among decaying solutions such that

(2.3) $$u(r;\alpha_1) = O\left(r^{2-n}\exp(-r^2/4)\right) \text{ as } r \to \infty.$$

Theorem 2 gives us more detailed structure of solutions to (IVP) with $\lambda = 1$ than the result established by Haraux and Weissler [HW].

3. Preliminary Results

In this section, we will give some fundamental properties of solutions to (IVP).

Proposition 3.1. The following two conditions are equivalent:

(i) $u(r;\alpha) \in C([0,\infty)) \cap C^2((0,\infty))$ satisfies (IVP).

(ii) $u(r;\alpha) \in C([0,\infty))$ satisfies

(3.1) $$u(r;\alpha) = \alpha - \int_0^r dt \int_0^t (s/t)^{n-1} \exp\{(s^2 - t^2)/4\}(\lambda u + |u|^{p-1}u)ds.$$

Moreover, in both cases, the following properties holds;

(a) $u(r;\alpha)$ is decreasing in $[0,z)$, where $z := \inf\{r > 0\ ;\ u(r;\alpha) = 0\}$. (If $u(r;\alpha) > 0$ in $[0,\infty)$, then we put $z = \infty$.)

(b) $u(r;\alpha) \in C^2([0,\infty))$ and $u_r(0;\alpha) = 0$.

(c) $|u(r;\alpha)| \le C(1+r)^{-2\lambda}$ and $|u_r(r;\alpha)| \le C(1+r)^{-2\lambda-1}$ for all $r \ge 0$, where C depends boundedly on α.

Proof. We first show that (i) implies (ii). For this purpose, we begin with the proof of (a). First we note that the equation of (IVP) is equivalent to

(3.2) $$\{r^{n-1}\exp(r^2/4)u_r\}_r + r^{n-1}\exp(r^2/4)(\lambda u + |u|^{p-1}u) = 0.$$

Integrating (3.2) over $[\theta, r]$ leads to

(3.3) $$r^{n-1}\exp(r^2/4)u_r(r;\alpha) - \theta^{n-1}\exp(\theta^2/4)u_r(\theta;\alpha) = -\int_\theta^r s^{n-1}\exp(s^2/4)(\lambda u + |u|^{p-1}u)ds.$$

Since $s^{n-1}\exp(s^2/4)(\lambda u + |u|^{p-1}u) \in L^1(0,r)$, there exists $\lim_{\theta\to 0}\theta^{n-1}u_r(\theta;\alpha)$. Now we will prove $\lim_{r\to 0} r^{n-1}u_r(r;\alpha) = 0$ by contradiction. Suppose that

(3.4) $$\lim_{r\to 0} r^{n-1}u_r(r;\alpha) =: \eta > 0.$$

(We can also derive a contradiction in the case $\eta < 0$.) Let ε be any sufficiently small positive number. From (3.4), we can take sufficiently small $\delta(\varepsilon) > 0$ such that

$$r^{1-n}(\eta - \varepsilon) < u_r(r;\alpha) < r^{1-n}(\eta + \varepsilon) \tag{3.5}$$

for $r \in (0,\delta(\varepsilon))$. Integrating (3.5) from r to δ, we get

$$u(\delta;\alpha) - \frac{\eta+\varepsilon}{n-2}(r^{2-n} - \delta^{2-n}) < u(r;\alpha) < u(\delta;\alpha) - \frac{\eta-\varepsilon}{n-2}(r^{2-n} - \delta^{2-n});$$

which implies $\lim_{r\to 0} u(r;\alpha) = -\infty$. Since this is absurd, we get $\lim_{\theta\to 0}\theta^{n-1}u_r(\theta;\alpha) = 0$. Therefore, letting $\theta \to 0$ in (3.3), we obtain

$$u_r(r;\alpha) = -\int_0^r (s/r)^{n-1}\exp\{(s^2 - r^2)/4\}(\lambda u + |u|^{p-1}u)ds. \tag{3.6}$$

Thus as far as $u(r;\alpha)$ is positive, $u_r(r;\alpha)$ is negative; so that $u(r;\alpha)$ is decreasing in $[0,z)$.

Moreover, Integrating (3.6) over $[0,r]$ and using $u(0) = \alpha$, we get (3.1). Thus we have shown that (i) implies (ii). Conversely, it is readily seen that (ii) implies (i). Concerning the proofs of (b) and (c), see [W2] and [HW], respectively. Q.E.D.

Proposition 3.2. There exists a unique solution $u(r;\alpha) \in C^2([0,\infty))$ of (IVP).

Proof. By Proposition 3.1, it is sufficient to show the uniqueness and existence of solutions for (3.1). The uniqueness is easily proved by Gronwall's inequality. The existence is obtained as follows. For $0 \le r \le \delta$ with a suitably small $\delta > 0$, we use the successive approximation method to obtain the local existence. For $r > \delta$, we introduce

$$E(r) := \frac{1}{2}u_r(r;\alpha)^2 + \frac{\lambda}{2}u(r;\alpha)^2 + \frac{1}{p+1}|u(r;\alpha)|^{p+1}.$$

Differentiating $E(r)$, we obtain

$$E'(r) = -\left\{\frac{n-1}{r} + \frac{r}{2}\right\}u_r^{\,2} \le 0.$$

Thus, since $u(r:\alpha)$ and $u_r(r:\alpha)$ can never blow up, the global existence of $u(r:\alpha)$ for every $r > 0$ can be proved in the standard manner. Q.E.D.

4. The Classification Theorem by Yanagida and Yotsutani

In this section, for the purpose to prove Theorems 1 and 2, we will explain *the classification theorem by Yanagida and Yotsutani* (see [YY2] or [Y]) for the following initial value problem

$$(4.1)\qquad \begin{cases} (g(r)u_r)_r + g(r)K(r)(u^+)^p = 0,\ r>0, \\ u(0)=\alpha>0, \end{cases}$$

where $u^+ = \max\{u,0\}$. We suppose that $g(r)$ and $K(r)$ satisfy

$$(g)\qquad \begin{cases} g(r)\in C^1([0,\infty)); \\ g(r)>0 \text{ in } (0,\infty); \\ 1/g(r)\notin L^1(0,1); \\ 1/g(r)\in L^1(1,\infty), \end{cases}$$

and

$$(K)\qquad \begin{cases} K(r)\in C(0,\infty); \\ K(r)\ge 0 \text{ and } K(r)\not\equiv 0 \text{ in } (0,\infty); \\ h(r)K(r)\in L^1(0,1); \\ h(r)\{h(r)/g(r)\}^p K(r)\in L^1(1,\infty), \end{cases}$$

where

$$h(r) := g(r)\int_r^\infty \{1/g(s)\}ds.$$

Moreover, define the following functions

$$G(r) := \frac{2}{p+1}g(r)h(r)K(r) - \int_0^r g(s)K(s)ds,$$

$$H(r) := \frac{2}{p+1}h(r)^2\left\{\frac{h(r)}{g(r)}\right\}^p K(r) - \int_r^\infty h(s)\left\{\frac{h(s)}{g(s)}\right\}^p K(s)ds,$$

and set

$$r_G := \inf\{r\in(0,\infty);\ G(r)<0\},\quad r_H := \sup\{r\in(0,\infty);\ H(r)<0\}.$$

Remark 4.1. We can show that (4.1) has a unique solution $u(r;\alpha)$ for each $\alpha>0$ under the first, second and third conditions in (K).

Now we will state their result.

Theorem 4.1. ([YY2]) Suppose that $G(r)\not\equiv 0$ in $[0,\infty)$. Let $u(r;\alpha)$ be the solution of (4.1).
(a) If $r_G=\infty$ (i.e., $G(r)\geq 0$ in $(0,\infty)$), then $u(r;\alpha)$ is a crossing solution for every $\alpha>0$.
(b) If $r_G<\infty$ and $r_H=0$ (i.e., $H(r)\geq 0$ in $(0,\infty)$), then $u(r;\alpha)$ is a decaying solution with $\lim_{r\to\infty}\{g(r)/h(r)\}u(r;\alpha)=\infty$ for every $\alpha>0$.
(c) If $0<r_H\leq r_G<\infty$, then there exists a unique positive number α_f such that $u(r;\alpha)$ is a crossing solution for every $\alpha\in(\alpha_f,\infty)$, and a decaying solution with $\lim_{r\to\infty}\{g(r)/h(r)\}u(r;\alpha)=\infty$ for every $\alpha\in(0,\alpha_f)$. Moreover, if $\alpha=\alpha_f$, then $u(r;\alpha)$ is a decaying solution with $0<\lim_{r\to\infty}\{g(r)/h(r)\}u(r;\alpha)<\infty$, which means that $u(r;\alpha_f)$ is the most rapidly decaying solution among decaying solutions.

Remark 4.2. If $G(r)\equiv 0$ in $[0,\infty)$, then for every $\alpha>0$, $u(r;\alpha)$ is a decaying solution with $0<\lim_{r\to\infty}\{g(r)/h(r)\}u(r;\alpha)<\infty$.

5. Proof of Theorem 1

In this section, we will study the following initial value problem

$$\text{(5.1)}\qquad \begin{cases} u_{rr}+\dfrac{n-1}{r}u_r+\dfrac{r}{2}u_r+(u^+)^p=0, & r>0,\\ u(0)=\alpha>0, \end{cases}$$

where $u^+=\max\{u,0\}$. The equation of (5.1) is equivalent to

$$\{r^{n-1}\exp(r^2/4)u_r\}_r+r^{n-1}\exp(r^2/4)(u^+)^p=0.$$

If we put $g(r):=r^{n-1}\exp(r^2/4)$ and $K(r):=1$ in (4.1), then it is easily seen that $g(r)$ and $K(r)$ satisfy (g) and (K), respectively. Moreover, we obtain

$$G(r)=2(p+1)^{-1}r^{2n-2}\exp(r^2/2)\int_r^\infty s^{1-n}\exp(-s^2/4)ds-\int_0^r s^{n-1}\exp(s^2/4)ds,$$

$$H(r)=2(p+1)^{-1}r^{2n-2}\exp(r^2/2)\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}^{p+2}$$
$$-\int_r^\infty s^{n-1}\exp(s^2/4)\left\{\int_s^\infty t^{1-n}\exp(-t^2/4)dt\right\}^{p+1}ds.$$

After some calculations,

(5.2) $G'(r)=2(p+1)^{-1}r^{n-1}\exp(r^2/4)\{\Phi(r)-(p+3)/2\}\equiv\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}^{-p-1}H'(r)$,

where

$$\Phi(r):=\{2(n-1)+r^2\}r^{n-2}\exp(r^2/4)\int_r^\infty s^{1-n}\exp(-s^2/4)ds. \tag{5.3}$$

In order to apply Theorem 4.1, we must know the location of r_G and r_H. For this purpose, we will investigate the profiles of $G(r)$ and $H(r)$. In view of (5.2), it is important to study $\Phi(r)$. First we obtain the following lemma.

Lemma 5.1.

(i) $\lim_{r\to 0}\Phi(r)=2(n-1)/(n-2)$.

(ii) $\Phi(r)=2-4r^{-2}+o(r^{-2})$ as $r\to\infty$.

(iii) There exists a unique number $r_0\in\left(0,\sqrt{6(n-1)}\right)$ such that $\Phi(r)$ is decreasing in $[0,r_0)$ and increasing in (r_0,∞). Moreover, $\Phi(r_0)<2$.

Proof. (i) By l'Hospital's theorem,

$$\begin{aligned}\lim_{r\to 0}\Phi(r)&=\lim_{r\to 0}\{2(n-1)+r^2\}r^{n-2}\exp(r^2/4)\int_r^\infty s^{1-n}\exp(-s^2/4)ds\\&=\lim_{r\to 0}\frac{\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}_r}{\left\{\left[\{2(n-1)+r^2\}r^{n-2}\right]^{-1}\right\}_r}\\&=\lim_{r\to 0}\frac{4(n-1)^2+4(n-1)r^2+r^4}{2(n-1)(n-2)+nr^2}=\frac{2(n-1)}{n-2}.\end{aligned}$$

(ii) Integrating by parts, we obtain

$$\begin{aligned}&\int_r^\infty s^{1-n}\exp(-s^2/4)ds\\&=2r^{-n}\exp(-r^2/4)-2n\int_r^\infty s^{-1-n}\exp(-s^2/4)ds\\&=2r^{-n}\exp(-r^2/4)-4nr^{-n-2}\exp(-r^2/4)+4n(n+2)\int_r^\infty s^{-3-n}\exp(-s^2/4)ds.\end{aligned}\tag{5.4}$$

Thus we get

$$\Phi(r)=2-4r^{-2}-8n(n-1)r^{-4}+4n(n+2)\{2(n-1)+r^2\}r^{n-2}\exp(r^2/4)\int_r^\infty s^{-3-n}\exp(-s^2/4)ds,$$

which implies (ii).

(iii) From (ii), $\Phi(r)$ is increasing for sufficiently large r and converges to 2. Moreover, since $2(n-1)/(n-2)>2$, $\Phi(r)$ must have a local minimum at some $r_0 \in (0,\infty)$, and it is smaller than 2. We will show that there are no other critical points of $\Phi(r)$. By direct calculations,

(5.5) $$\Phi'(r) = -2(n-1)r^{-1} - r + \{2(n-1)(n-2) + (2n-1)r^2 + r^4/2\}r^{n-3}\exp(r^2/4)\int_r^\infty s^{1-n}\exp(-s^2/4)ds,$$

(5.6) $$\Phi''(r) = -2(n-1)(n-3)r^{-2} - 2n - r^2/2 + \{2(n-1)(n-2)(n-3) + 3(n-1)^2r^2 + 3nr^4/2 + r^6/4\}r^{n-4}\exp(r^2/4)\int_r^\infty s^{1-n}\exp(-s^2/4)ds.$$

Suppose that there exists a positive number $\tilde{r}$ such that $\Phi'(\tilde{r}) = 0$. It follows from (5.5) that

(5.7) $$\tilde{r}^{n-2}\exp(\tilde{r}^2/4)\int_{\tilde{r}}^\infty s^{1-n}\exp(-s^2/4)ds = \frac{2\tilde{r}^2 + 4(n-1)}{\tilde{r}^4 + 2(2n-1)\tilde{r}^2 + 4(n-1)(n-2)}.$$

Combining (5.6) and (5.7) leads to

(5.8) $$\Phi''(\tilde{r}) = \frac{-4\left(\tilde{r} + \sqrt{6(n-1)}\right)\left(\tilde{r} - \sqrt{6(n-1)}\right)}{\tilde{r}^4 + 2(2n-1)\tilde{r}^2 + 4(n-1)(n-2)}.$$

From (5.8), $\Phi''(\tilde{r}) > 0$ if $\tilde{r} \in \left(0, \sqrt{6(n-1)}\right)$ and $\Phi''(\tilde{r}) < 0$ if $\tilde{r} \in \left(\sqrt{6(n-1)}, \infty\right)$. Therefore, if $\Phi(r)$ has a critical point, then it must be a local minimum in $\left(0, \sqrt{6(n-1)}\right)$ and a local maximum in $\left(\sqrt{6(n-1)}, \infty\right)$. This result says that there exist at most one local minimum and one local maximum since a local maximum cannot exist in $\left(0, \sqrt{6(n-1)}\right)$ and a local minimum cannot exist in $\left(\sqrt{6(n-1)}, \infty\right)$. We have already known that $\Phi(r)$ has a local minimum, and now we will show that $\Phi(r)$ cannot have a local maximum. In fact, suppose that there exists a local maximum. Then $\Phi(r)$ decreases for large r. But it is impossible, because (ii) of this lemma means that $\Phi(r)$ increasingly converges to 2. Thus we finish the proof of (iii). (See Fig.1.)

Q.E.D.

From Lemma 5.1, since $2 < (p+3)/2 < 2(n-1)/(n-2)$ if $1 < p < (n+2)/(n-2)$, there exists a unique number $r_* \in (0,\infty)$ such that $\Phi(r) > (p+3)/2$ in $(0, r_*)$, $\Phi(r_*) = (p+3)/2$ and $\Phi(r) < (p+3)/2$ in (r_*, ∞) (see Fig.2). Moreover, since $(p+3)/2 \geq 2(n-1)/(n-2)$ if $p \geq (n+2)/(n-2)$, $\Phi(r) \leq (p+3)/2$ in $[0,\infty)$. Therefore, in view of the expressions of (5.2), we get the following lemma.

Lemma 5.2.

(i) If $p \geq (n+2)/(n-2)$, then $G(r)$ and $H(r)$ are decreasing in $[0,\infty)$.

(ii) If $1<p<(n+2)/(n-2)$, then there exists a unique number $r_* \in (0,\infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0,r_*)$ and decreasing in (r_*,∞).

The behaviors of $G(r)$ and $H(r)$ near $r=0$ and $r=\infty$ are shown by the following result.

Lemma 5.3.

(i) $\lim_{r\to\infty} G(r) = -\infty$.

(ii) $\lim_{r\to 0} G(r) = 0$.

(iii) $\liminf_{r\to\infty} H(r) \geq 0$.

(iv) If $1<p<(n+2)/(n-2)$, then $\limsup_{r\to 0} H(r) < 0$.

Remark 5.1. If $p \geq (n+2)/(n-2)$, then $H(r) \geq 0$ and $H(r) \not\equiv 0$ in $[0,\infty)$ from Lemma 5.2 (i) and Lemma 5.3 (iii).

Proof. (i) By Lemma 5.1, $\{\Phi(r)-(p+3)/2\}$ is finitely negative for sufficiently large r and does not decay to zero as $r\to\infty$. Moreover, since $\lim_{r\to\infty} r^{n-1}\exp(r^2/4) = +\infty$, we obtain $\lim_{r\to\infty} G'(r) = -\infty$. Therefore, we get (i).

(ii) Since $\lim_{r\to 0}\int_0^r s^{n-1}\exp(s^2/4)ds = 0$, it is sufficient to show

$$\lim_{r\to 0} r^{2n-2}\exp(r^2/2)\int_r^\infty s^{1-n}\exp(-s^2/4)ds = 0.$$

In fact, by l'Hospital's theorem,

$$\lim_{r\to 0}\frac{\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}_r}{\left(r^{2-2n}\right)_r} = \lim_{r\to 0}\frac{r^{1-n}\exp(-r^2/4)}{(2n-2)r^{1-2n}} = 0.$$

(iii)
$$H(r) > -\int_r^\infty s^{n-1}\exp(s^2/4)\left\{\int_s^\infty t^{1-n}\exp(-t^2/4)dt\right\}^{p+1} ds$$
$$> -(n-2)^{-p-1}\int_r^\infty s^{n-1+(2-n)(p+1)}\exp(-ps^2/4)ds.$$

Therefore, we get

$$\liminf_{r\to\infty} H(r) \geq -(n-2)^{-p-1} \lim_{r\to\infty} \int_r^\infty s^{n-1+(2-n)(p+1)} \exp(-ps^2/4) ds = 0.$$

(iv) Let $p \in (1,(n+2)/(n-2))$. Assume ε be any sufficiently small positive number with $\varepsilon < \{(n+2)-(n-2)p\}/(n-2)(p+1)$ and fix ρ such that $\exp\{-(p+1)\rho^2/4\} > 1-\varepsilon$. Then for $0 < r < \rho$,

$$(5.9) \quad H(r) < \frac{2}{p+1} r^{2n-2} \exp\left(\frac{r^2}{2}\right) \left\{ \int_r^\infty s^{1-n} \exp\left(-\frac{s^2}{4}\right) ds \right\}^{p+2}$$

$$- \int_r^\rho s^{n-1} \exp\left(\frac{s^2}{4}\right) \left\{ \int_s^\rho t^{1-n} \exp\left(-\frac{t^2}{4}\right) dt \right\}^{p+1} ds$$

$$< \frac{2}{p+1} r^{2n-2} \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)r^2}{4}\right\} \frac{1}{(n-2)^{p+2}} r^{(2-n)(p+2)}$$

$$- \int_r^\rho s^{n-1} \exp\left(\frac{s^2}{4}\right) \exp\left\{-\frac{(p+1)\rho^2}{4}\right\} \frac{1}{(n-2)^{p+1}} s^{(2-n)(p+1)} \left\{1 - \left(\frac{s}{\rho}\right)^{n-2}\right\}^{p+1} ds.$$

First we consider the case $2 < p < 5$ for $n = 3$ and $1 < p < (n+2)/(n-2)$ for $n \geq 4$. Since $p+1 < 6$ and $2-(n-2)p < 0$, we obtain

$$H(r) < \frac{2}{(p+1)(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right)$$

$$- \frac{1}{(n-2)^{p+1}} \exp\left(\frac{r^2}{4}\right) \exp\left\{-\frac{(p+1)\rho^2}{4}\right\} \int_r^\rho s^{1-(n-2)p} \left\{1 - \left(\frac{s}{\rho}\right)^{n-2}\right\}^6 ds$$

$$< \frac{2}{(p+1)(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right)$$

$$+ \frac{1}{\{2-(n-2)p\}(n-2)^{p+1}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right)(1-\varepsilon) + o\left(r^{2-(n-2)p}\right)$$

$$= -\frac{(n+2)-(n-2)p-\varepsilon(n-2)(p+1)}{(p+1)\{(n-2)p-2\}(n-2)^{p+2}} r^{2-(n-2)p} \exp\left(\frac{r^2}{4}\right) + o\left(r^{2-(n-2)p}\right);$$

so that

$$\lim_{r\to 0} H(r) = -\infty.$$

In the case $p = 2$ for $n = 3$, it follows from the last inequality of (5.9) that

$$H(r) < 2\exp(-r^2/2)/3 - \exp(r^2/4)\exp(-3\rho^2/4) \int_r^\rho s^{-1} \{1-(s/\rho)\}^3 ds$$

$$< 2/3 - (1-\varepsilon)(\log\rho - \log r + O(1))$$

$$= (1-\varepsilon)\log r + O(1)$$

Then we arrive at the same result as before. It remains to discuss the case $1<p<2$ for $n=3$. Since $p+1<3$, we get

$$\begin{aligned}H(r)&<\frac{2}{p+1}r^{2-p}\exp\left(\frac{r^2}{4}\right)\exp\left\{-\frac{(p+1)r^2}{4}\right\}-\exp\left(\frac{r^2}{4}\right)\exp\left\{-\frac{(p+1)\rho^2}{4}\right\}\int_r^\rho s^{1-p}\left(1-\frac{s}{\rho}\right)^3 ds\\&<\frac{2}{p+1}r^{2-p}\exp\left(\frac{r^2}{4}\right)-\exp\left(\frac{r^2}{4}\right)(1-\varepsilon)\int_r^\rho\left\{s^{1-p}-3\frac{s^{2-p}}{\rho}+3\frac{s^{3-p}}{\rho^2}-\frac{s^{4-p}}{\rho^3}\right\}ds\\&=\left[\left\{\frac{2}{p+1}+\frac{1-\varepsilon}{2-p}\right\}r^{2-p}+o\left(r^{2-p}\right)\right]\exp\left(\frac{r^2}{4}\right)-\frac{6(1-\varepsilon)}{(2-p)(3-p)(4-p)(5-p)}\exp\left(\frac{r^2}{4}\right)\rho^{2-p}\end{aligned}$$

from (5.9). Thus we obtain

$$\limsup_{r\to0}H(r)\le-\frac{6(1-\varepsilon)}{(2-p)(3-p)(4-p)(5-p)}\rho^{2-p}<0.$$

Q.E.D.

Proof of Theorem 1. From Lemmas 5.2 and 5.3, we can draw the graphs of $G(r)$ and $H(r)$. Then we obtain $r_G=0(<\infty)$ and $r_H=0$ in the case $p\ge(n+2)/(n-2)$ (see Fig.3) and $0<r_H<r_G<\infty$ in the case $1<p<(n+2)/(n-2)$ (see Fig.4). So we can apply Theorem 4.1 to show Theorem 1.

We will show (2.1). From Theorem 4.1, there exists a positive finite number β such that

$$\lim_{r\to\infty}\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}^{-1}u(r;\alpha_0)=\beta.$$

Moreover, by using the fact that $\left\{\int_r^\infty s^{1-n}\exp(-s^2/4)ds\right\}^{-1}u(r;\alpha_0)$ is increasing in $[0,\infty)$, it follows from (5.4) that

$$\begin{aligned}u(r;\alpha_0)&<\beta\int_r^\infty s^{1-n}\exp(-s^2/4)ds\\&=2\beta\left\{r^{-n}\exp(-r^2/4)-2nr^{-n-2}\exp(-r^2/4)+2n(n+2)\int_r^\infty s^{-3-n}\exp(-s^2/4)ds\right\}.\end{aligned}$$

This implies (2.1). Q.E.D.

6. Proof of Theorem 2

In this section, we will study (IVP) with $\lambda=1$. Put

$$u(r):=v(r)\varphi(r),$$

then the equation of (IVP) is rewritten as

$$v_{rr}+\left(2\frac{\varphi_r}{\varphi}+\frac{n-1}{r}+\frac{r}{2}\right)v_r+|\varphi|^{p-1}|v|^{p-1}v+\left\{\frac{\varphi_{rr}}{\varphi}+\left(\frac{n-1}{r}+\frac{r}{2}\right)\frac{\varphi_r}{\varphi}+\lambda\right\}v=0.$$

Therefore, if we take $\varphi(r)$ which satisfies the following initial value problem

$$\text{(6.1)}\qquad \begin{cases}\varphi_{rr}+\left(\frac{n-1}{r}+\frac{r}{2}\right)\varphi_r+\lambda\varphi=0, & r>0,\\ \varphi(0)=1, \quad \varphi_r(0)=0,\end{cases}$$

then $v(r)$ must satisfy

$$\begin{cases}v_{rr}+\left(2\frac{\varphi_r}{\varphi}+\frac{n-1}{r}+\frac{r}{2}\right)v_r+|\varphi|^{p-1}|v|^{p-1}v=0, & r>0,\\ v(0)=\alpha>0.\end{cases}$$

In the special case $\lambda=1$, it is possible to express the $C^2[0,\infty)$-solution of (6.1) by

$$\varphi(r)=(n-2)r^{2-n}\exp(-r^2/4)\int_0^r s^{n-3}\exp(s^2/4)ds.$$

Note that $\varphi(r)>0$ in $[0,\infty)$. In order to know the structure of solutions to (IVP) with $\lambda=1$, we have only to verify whether $v(r;\alpha)$ has a zero or not. In this section, we will mainly study

$$\text{(6.2)}\qquad \begin{cases}v_{rr}+\left(2\frac{\varphi_r}{\varphi}+\frac{n-1}{r}+\frac{r}{2}\right)v_r+\varphi^{p-1}(v^+)^p=0, & r>0,\\ v(0)=\alpha>0.\end{cases}$$

The equation of (6.2) is equivalent to

$$\left\{r^{n-1}\exp(r^2/4)\varphi^2 v_r\right\}_r+r^{n-1}\exp(r^2/4)\varphi^2\cdot\varphi^{p-1}(v^+)^p=0;$$

to which Theorem 4.1 is applicable. In fact, we obtain following proposition.

Proposition 6.1. Put $g(r):=r^{n-1}\exp(r^2/4)\varphi^2$ and $K(r):=\varphi^{p-1}$. Then $g(r)$ and $K(r)$ satisfy (g) and (K), respectively.

Proof. We can readily see that $g(r)$ and $K(r)$ satisfy $(g)_1$, $(g)_2$, $(K)_1$ and $(K)_2$, where $(g)_i$ and $(K)_i$ mean the i-th condition of (g) and (K), respectively. Moreover,

$(g)_3$ Since $1/g(r)=r^{1-n}+o(r^{1-n})$ as $r\to 0$, we get $1/g(r)\notin L^1(0,1)$.

$(g)_4$ Integrating by parts, we obtain

$$\int_0^r s^{n-3}\exp(s^2/4)ds=2r^{n-4}\exp(r^2/4)-4(n-4)r^{n-6}\exp(r^2/4)$$
$$+4(n-4)(n-6)\int_1^r s^{n-7}\exp(s^2/4)ds+\int_0^1 s^{n-3}\exp(s^2/4)ds+(4n-18)e^{1/4};$$

so that

(6.3) $$\varphi(r)=2(n-2)r^{-2}-4(n-2)(n-4)r^{-4}+o(r^{-4}) \text{ as } r\to\infty.$$

From (6.3), since

$$1/g(r)=r^{5-n}\exp(-r^2/4)(1+o(1))/4(n-2)^2 \text{ as } r\to\infty,$$

we have $1/g(r)\in L^1(1,\infty)$.

$(K)_3$ Note that

$$\begin{aligned}h(r)&=g(r)\int_r^\infty\{1/g(s)\}ds\\ &=r^{3-n}\exp(-r^2/4)\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^2\left[\int_r^\infty s^{n-3}\exp(s^2/4)\left\{\int_0^s t^{n-3}\exp(t^2/4)dt\right\}^{-2}ds\right]\\ &=r^{3-n}\exp(-r^2/4)\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^2\int_\tau^\infty(1/T^2)dT\\ &=r^{3-n}\exp(-r^2/4)\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\}=r\varphi(r)/(n-2),\end{aligned}$$

where $\tau:=\int_0^r t^{n-3}\exp(t^2/4)dt$. So we readily obtain

$$h(r)K(r)=r\varphi(r)^p/(n-2)\in L^1(0,1).$$

Condition $(K)_4$ is readily seen by

$$h(r)\{h(r)/g(r)\}^pK(r)=r^{1+(2-n)p}\exp(-pr^2/4)/(n-2)^{p+1}\in L^1(1,\infty).$$ Q.E.D.

Now we obtain

$$G(r)=(n-2)^{p+1}\left[\frac{2}{p+1}r^{4-n+(2-n)p}\exp\left\{-\frac{(p+1)r^2}{4}\right\}\left\{\int_0^r s^{n-3}\exp\left(\frac{s^2}{4}\right)ds\right\}^{p+2}\right.$$
$$\left.-\int_0^r s^{1+(2-n)p}\exp\left(-\frac{ps^2}{4}\right)\left\{\int_0^s t^{n-3}\exp\left(\frac{t^2}{4}\right)dt\right\}^{p+1}ds\right],$$

$$H(r)=\frac{1}{(n-2)^{p+1}}\left[\frac{2}{p+1}r^{4-n+(2-n)p}\exp\left\{-\frac{(p+1)r^2}{4}\right\}\int_0^r s^{n-3}\exp\left(\frac{s^2}{4}\right)ds\right.$$
$$\left.-\int_r^\infty s^{1+(2-n)p}\exp\left(-\frac{ps^2}{4}\right)ds\right].$$

Differentiating $G(r)$ and $H(r)$, we get

(6.4) $$H'(r)=\frac{2}{(p+1)(n-2)^{p+1}}r^{1+(2-n)p}\exp\left(-\frac{pr^2}{4}\right)\left\{\Psi(r)-\frac{p+3}{2}\right\}=\left\{\int_r^\infty\frac{1}{g(s)}ds\right\}^{p+1}G'(r),$$

where

$$(6.5)\qquad \Psi(r):=(p+3)-\frac{1}{n-2}\varphi(r)\left[\{(n-2)p+n-4\}+\frac{p+1}{2}r^2\right]$$

by recalling the expression of $\varphi(r)=(n-2)r^{2-n}\exp(-r^2/4)\int_0^r s^{n-3}\exp(s^2/4)ds$.

In order to prove Theorem 2, we will use the same argument as in Section 5. First, we will investigate the profile of $\Psi(r)$.

Lemma 6.1.

(i) $\lim_{r\to 0}\Psi(r)=2(n-1)/(n-2)$.

(ii) $\Psi(r)=2-4pr^{-2}+o(r^{-2})$ as $r\to\infty$.

(iii) There exists a unique number $r_1\in\left(\sqrt{2(p+2)\{(n-2)p+n-4\}/\{p(p+1)\}},\infty\right)$ such that $\Psi(r)$ is decreasing in $[0,r_1)$ and increasing in (r_1,∞). Moreover, $\Psi(r_1)<2$.

Proof. (i) Since $\lim_{r\to 0}\varphi(r)=1$ and $\lim_{r\to 0}r^2\varphi(r)=0$, the conclusion easily follows.

(ii) Using (6.3) for sufficiently large r, we obtain

$$\begin{aligned}\Psi(r)&=(p+3)-\{2r^{-2}-4(n-4)r^{-4}+o(r^{-4})\}\left[\{(n-2)p+n-4\}+\frac{p+1}{2}r^2\right]\\&=2-4pr^{-2}+o(r^{-2}).\end{aligned}$$

(iii) Since $\Psi(r)$ increasingly converges to 2 from (ii) and $2(n-1)/(n-2)>2$, $\Psi(r)$ must have a local minimum at some $r_1\in(0,\infty)$ and $\Psi(r_1)<2$. We will show that there are no other critical points of $\Psi(r)$. Direct calculations yield

$$\begin{aligned}(6.6)\quad \Psi'(r)=&-\{(n-2)p+n-4\}r^{-1}-(p+1)r/2\\&+\left[(n-2)\{(n-2)p+n-4\}+\{(n-3)p+n-4\}r^2+(p+1)r^4/4\right]\\&\times r^{1-n}\exp(-r^2/4)\int_0^r s^{n-3}\exp(s^2/4)ds,\end{aligned}$$

$$\begin{aligned}(6.7)\quad \Psi''(r)=&(n-1)\{(n-2)p+n-4\}r^{-2}+\{(2n-7)p+2n-9\}/2+(p+1)r^2/4\\&+\left[(1-n)(n-2)\{(n-2)p+n-4\}+\{(-3n^2+16n-22)p-3n^2+20n-32\}r^2/2\right.\\&\left.+\{(-3n+11)p-3n+13\}r^4/4-(p+1)r^6/8\right]r^{-n}\exp(-r^2/4)\int_0^r s^{n-3}\exp(s^2/4)ds.\end{aligned}$$

Suppose that there exists a positive number $\hat{r}$ such that $\Psi'(\hat{r}) = 0$. Then by (6.6), we have

$$\text{(6.8)}\quad \hat{r}^{-n}\exp\left(-\hat{r}^2/4\right)\int_0^{\hat{r}} s^{n-3}\exp\left(s^2/4\right)ds$$
$$= \frac{\{(n-2)p+n-4\}+(p+1)\hat{r}^2/2}{(n-2)\{(n-2)p+n-4\}\hat{r}^2+\{(n-3)p+n-4\}\hat{r}^4+(p+1)\hat{r}^6/4}.$$

When $n=3$, the right hand side of (6.8) is non-positive for some $\hat{r}$. But the left hand side of (6.8) is positive for every $\hat{r}$. Therefore, for $n=3$, we observe that $\Psi(r)$ cannot have any critical points for r satisfying

$$(p-1)r^2 - r^4 + (p+1)r^6/4 \le 0.$$

Combining (6.7) and (6.8) leads to

$$\text{(6.9)}\qquad \Psi''(\hat{r}) = \frac{-2(p+2)\{(n-2)p+n-4\}+p(p+1)\hat{r}^2}{(n-2)\{(n-2)p+n-4\}+\{(n-3)p+n-4\}\hat{r}^2+(p+1)\hat{r}^4/4}.$$

Let $r_p := \sqrt{2(p+2)\{(n-2)p+n-4\}/\{p(p+1)\}}$. From (6.9), $\Psi''(\hat{r})<0$ for $\hat{r}\in(0,r_p)$ and $\Psi''(\hat{r})>0$ for $\hat{r}\in(r_p,\infty)$. Therefore, if $\Psi(r)$ has a critical point, then it must be a local maximum in $(0,r_p)$ and a local minimum in (r_p,∞). This result says that there exists at most one local maximum and one local minimum since a local minimum cannot exist in $(0,r_p)$ and a local maximum cannot exist in (r_p,∞). Moreover, we will evaluate the critical value for $\Psi(r)$. Combining (6.5) and (6.8), we get

$$\Psi(\hat{r}) = \frac{(p+1)\hat{r}^4/2-\{p^2-(2n-7)p-2n+8\}\hat{r}^2+2(n-1)\{(n-2)p+n-4\}}{(p+1)\hat{r}^4/4+\{(n-3)p+n-4\}\hat{r}^2+(n-2)\{(n-2)p+n-4\}}.$$

Define

$$\psi(r) := \frac{(p+1)r^4/2-\{p^2-(2n-7)p-2n+8\}r^2+2(n-1)\{(n-2)p+n-4\}}{(p+1)r^4/4+\{(n-3)p+n-4\}r^2+(n-2)\{(n-2)p+n-4\}} \quad \text{in } [0,\infty).$$

Then $\psi(r)$ satisfies $\psi(0)=2(n-1)/(n-2)$, $\lim_{r\to\infty}\psi(r)=2$ and

$$\text{(6.10)}\quad \psi'(r)$$
$$= \frac{p(p+1)^2 r\left[r^4-4\{(n-2)p+n-4\}r^2/p(p+1)-4(p+2)\{(n-2)p+n-4\}^2/p(p+1)^2\right]/2}{\left[(p+1)r^4/4+\{(n-3)p+n-4\}r^2+(n-2)\{(n-2)p+n-4\}\right]^2}$$

$$= \frac{p(p+1)^2 r\left[r^2+2\{(n-2)p+n-4\}/(p+1)\right](r+r_p)(r-r_p)/2}{\left[(p+1)r^4/4+\{(n-3)p+n-4\}r^2+(n-2)\{(n-2)p+n-4\}\right]^2}.$$

Since $2\{(n-2)p+n-4\}>0$ for $n \geq 3$, it follows from (6.10) that $\psi(r)$ is decreasing in $(0,r_p)$ and increasing in (r_p,∞). Therefore, $\Psi(r)$ has at most one local maximum in $(0,r_p)$, and it is smaller than $2(n-1)/(n-2)$. But this is impossible from (i) of Lemma 6.1. Therefore, $\Psi(r)$ does not have any local maximum. Thus we can finish the proof of (iv). Q.E.D.

Correspondingly to Lemma 5.2, we obtain the following lemma.

Lemma 6.2.

(i) If $p \geq (n+2)/(n-2)$, then $G(r)$ and $H(r)$ are decreasing in $[0,\infty)$.

(ii) If $1<p<(n+2)/(n-2)$, then there exists a unique number $r_{**} \in (0,\infty)$ such that $G(r)$ and $H(r)$ are increasing in $[0,r_{**})$ and decreasing in (r_{**},∞).

The behaviors of $G(r)$ and $H(r)$ near $r=0$ and $r=\infty$ are given as follows.

Lemma 6.3.

(i) $\lim_{r\to\infty} G(r) = -\infty$.

(ii) $\lim_{r\to 0} G(r) = 0$.

(iii) $\liminf_{r\to\infty} H(r) \geq 0$.

(iv) If $1<p<(n+2)/(n-2)$, then $\limsup_{r\to 0} H(r) < 0$.

Remark 6.1. If $p \geq (n+2)/(n-2)$, then $H(r) \geq 0$ and $H(r) \not\equiv 0$ in $[0,\infty)$ from Lemma 6.2 (i) and Lemma 6.3 (iii).

Proof. (i) Note that (6.4) can be rewritten as

$$G'(r) = \frac{2}{p+1}\left(r^2\varphi(r)\right)^{p+1} r^{n-2p-3}\exp\left(\frac{r^2}{4}\right)\left\{\Psi(r) - \frac{p+3}{2}\right\}.$$

By Lemma 6.1, $\{\Psi(r)-(p+3)/2\}$ is finitely negative for sufficiently large r and does not converge to zero as $r \to \infty$. Moreover, since $\lim_{r\to\infty} r^2\varphi(r) = 2$ from (6.3) and $\lim_{r\to\infty} r^{n-2p-3}\exp(r^2/4) = \infty$, we get (i).

(ii) Since $\lim_{r\to 0}\int_0^r s^{1+(2-n)p}\exp(-ps^2/4)\left\{\int_0^s t^{n-3}\exp(t^2/4)dt\right\}^{p+1}ds = 0$, it is sufficient to prove

$$\lim_{r\to 0} r^{4-n+(2-n)p}\exp\{-(p+1)r^2/4\}\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^{p+2} = 0;$$

which comes from the identity

$$r^{4-n+(2-n)p}\exp\{-(p+1)r^2/4\}\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^{p+2} = r^n\exp(r^2/4)\varphi(r)^{p+2}/(n-2)^{p+2}.$$

(iii) The assertion is readily seen from the following inequality

$$H(r) > -(n-2)^{-p-1}\int_r^\infty s^{1+(2-n)p}\exp(-ps^2/4)ds.$$

(iv) Let $p \in (1,(n+2)/(n-2))$. Assume ε be any sufficiently small positive number with $\varepsilon < \{(n+2)-(n-2)p\}/(n-2)(p+1)$ and fix ρ such that $\exp\{-(p+1)\rho^2/4\} > 1-\varepsilon$. Then for $0 < r < \rho$,

$$(6.11)\ H(r) < \frac{1}{(n-2)^{p+1}}\left[\frac{2}{p+1}r^{4-n+(2-n)p}\exp\left\{-\frac{(p+1)r^2}{4}\right\}\int_0^r s^{n-3}\exp\left(\frac{s^2}{4}\right)ds - \int_r^\rho s^{1+(2-n)p}\exp\left(-\frac{ps^2}{4}\right)ds\right]$$

$$< \frac{1}{(n-2)^{p+1}}\left[\frac{2}{(p+1)(n-2)}r^{2+(2-n)p}\exp\left(-\frac{pr^2}{4}\right) - \exp\left(-\frac{p\rho^2}{4}\right)\int_r^\rho s^{1+(2-n)p}ds\right].$$

First considering the case $2 < p < 5$ for $n = 3$ and $1 < p < (n+2)/(n-2)$ for $n \geq 4$, we obtain

$$H(r) < -\frac{(n+2)-(n-2)p-\varepsilon(n-2)(p+1)}{(p+1)\{(n-2)p-2\}(n-2)^{p+2}}r^{2-(n-2)p}\exp\left(\frac{r^2}{4}\right) + o\left(r^{2-(n-2)p}\right);$$

so that

$$\lim_{r\to 0} H(r) = -\infty.$$

In the case $p = 2$ for $n = 3$, observing that

$$H(r) < 2\exp(-r^2/2)/3 - \exp(-\rho^2/2)(\log\rho - \log r)$$
$$< (1-\varepsilon)\cdot\log r + O(1)$$

from (6.11), we arrive at the same result as before. Moreover, in the case $1 < p < 2$ for $n = 3$, we get

$$H(r) < \frac{1}{(n-2)^{p+1}}\left\{\frac{2}{p+1}r^{2-p}\exp(-pr^2/4) - \frac{1}{2-p}\exp(-p\rho^2/4)(\rho^{2-p} - r^{2-p})\right\}$$

from (6.11). Thus we obtain

$$\limsup_{r\to 0} H(r) \le -\frac{1}{(2-p)(n-2)^{p+1}} \exp\left(-\frac{p\rho^2}{4}\right)\rho^{2-p} < 0$$

since $2-p>0$. Q.E.D.

In the same way as the proof of Theorem 1, we obtain the following theorem.

Theorem 6.1. The structure of positive solutions to (6.2) is as follows.

(i) If $p \ge (n+2)/(n-2)$, then $v(r;\alpha)$ is a decaying solution for every $\alpha>0$.

(ii) If $1<p<(n+2)/(n-2)$, then there exists a unique positive number α_1 such that $v(r;\alpha)$ is a decaying solution for every $\alpha \in (0,\alpha_1]$ and a crossing solution for every $\alpha \in (\alpha_1,\infty)$. Moreover, $v(r;\alpha_1)$ is the most rapidly decaying solution among decaying solutions and there exists a positive finite number γ such that

$$\lim_{r\to\infty}\left\{(n-2)^2\int_0^r s^{n-3}\exp(s^2/4)ds\right\}v(r,\alpha_1)=\gamma .$$

Proof of Theorem 2. The structure of positive solutions to (IVP) with $\lambda=1$ is readily obtained by Theorem 6.1. We will show (2.3). Using the fact that $\left\{(n-2)^2\int_0^r s^{n-3}\exp(s^2/4)ds\right\}v(r,\alpha_1)$ is increasing in $[0,\infty)$, we get

$$v(r,\alpha_1)<\gamma\left\{(n-2)^2\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^{-1}.$$

Therefore, we have

$$\begin{aligned} u(r;\alpha_1) &= v(r;\alpha_1)\varphi(r) \\ &< \gamma\left\{(n-2)^2\int_0^r s^{n-3}\exp(s^2/4)ds\right\}^{-1}\cdot(n-2)r^{2-n}\exp(-r^2/4)\left\{\int_0^r s^{n-3}\exp(s^2/4)ds\right\} \\ &= (n-2)^{-1}\gamma r^{2-n}\exp(-r^2/4). \end{aligned}$$

This implies (2.3). Q.E.D.

7. Appendix

After this talk, I have obtained the following result on the structure of solutions to (IVP).

Theorem 7.1. Suppose that $0 \le \lambda \le (n-2)/2$. If $1 < p < (n+2)/(n-2)$, then there exists a unique positive number α_λ such that $u(r;\alpha)$ is a decaying solution for every $\alpha \in (0, \alpha_\lambda]$ and a crossing solution for every $\alpha \in (\alpha_\lambda, \infty)$. Moreover, $u(r;\alpha_\lambda)$ is the most rapidly decaying solution among decaying solutions.

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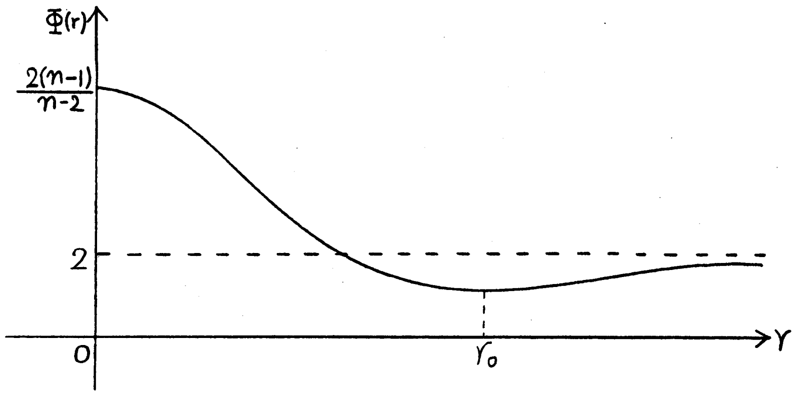

Fig. 1

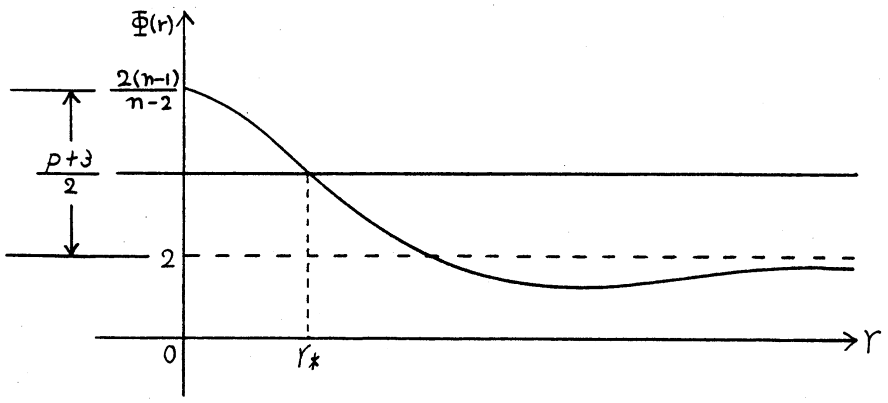

Fig. 2

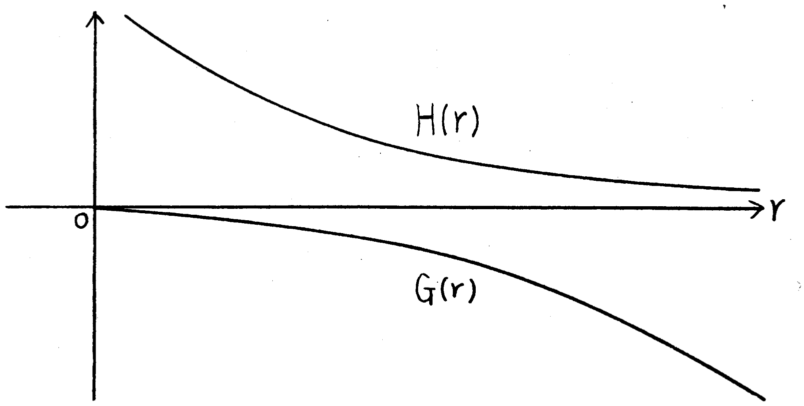

Fig. 3

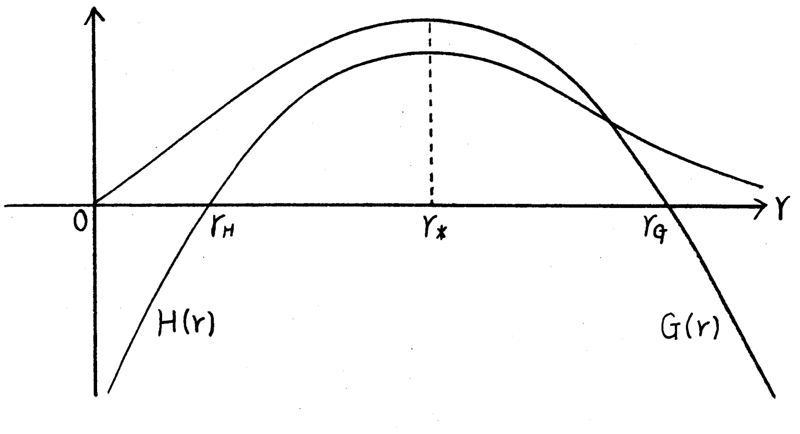

Fig. 4